Basins of Attraction in Two-Player Random Potential Games

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- Still, the question is ill-posed if we don't clarify the meaning of "drawing payoffs at random."
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- If payoffs are random, PNE may or may not exist.
- More precisely, their number is a random variable.
- It is easy to see that the expectation of this random variable is 1.

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- Powers (1990) proved that, as the number of actions diverges, the number of PNE converges in distribution to a Poisson(1).
- Both papers use the Chen-Stein method to prove their results.

Non-i.i.d. payoffs

- Rinott and S (2000) retained the assumption of independence for the random payoff vectors corresponding to different action profiles, but allowed for dependence of payoffs within the same action profile.
- They proved a phase transition in the correlation coefficient as either the number of players or the number of actions diverges.
- When correlation is negative, the number of pure Nash equilibria converges to 0, when the correlation is zero, it converges to Poisson(1), when the correlation is positive, it diverges and a central limit theorem holds.

Non-continuous distributions

- Amiet et al. (2021a) considered games with two actions for each player and with i.i.d. payoffs whose distribution is not necessarily continuous.
- They showed that several properties of these games depend on a unique parameter α, which represents the probability that two payoffs have a tie.
- In particular, as soon as $\alpha > 0$, the number of PNE diverges exponentially in the number of players N and the speed depends on α, but the number of strict PNE goes to zero.

Best-response dynamics

- Amiet et al. (2021a) studied also the asymptotic behavior of best-response dynamics (BRD) in this class of games.
- Interesting phase transitions appear.
- In particular,
  - When $0 \leq \alpha < 1/2$, for N large enough, any PNE is potentially reachable by BRD.
  - When $\alpha = 1/2$, with positive probability there exist PNE that are not reachable.
  - For $\alpha > 1/2$, the number of PNE that are not reachable grows to infinity with probability that approaches 1.

Best vs better

- Amiet et al. (2021b) considered two-player games with the same number of actions for each player, i.i.d. payoffs and continuous distributions.
- They compared the behavior of BRD and better-response dynamics (bRD).
- They showed that, asymptotically in the number of actions for each player, with probability 1, BRD does not converge to a PNE, whereas, with high probability, bRD converges to a PNE, whenever a PNE exists.

Other non-i.i.d. models

- Mimun et al.(2024) considered a class of two-player games that interpolates games with i.i.d. payoffs (and continuous distributions) and potential games.
- They considered asymptotic results as the number of actions of the two players diverge (not necessarily at the same speed).
- The results that they obtain depend on a single parameter $p \in [0, 1]$, the interpolation parameter.
- In particular, the number of PNE diverges, as soon as there is a positive weight on potential games.
- Moreover, they studied the random time a BRD needs to reach a PNE.

Ordinal potential games

- We let $[K] \coloneqq \{1, \dots, K\}$.
- We consider two-player normal-form ordinal games where each player $i \in \{\mathsf{A}, \mathsf{B}\}$ has the same action set $[K]$ and a preference relation $\prec_i$ over the outcomes of the game.
- These preferences are strict, i.e., for all pairs of outcomes Θ, Θ', either $\Theta \prec_i \Theta'$ or $\Theta' \prec_i \Theta$.
- A strategy profile (a^*, b^*) is a Nash equilibrium (NE) of the game if, for all $a, b \in K$,

$$(a, b^*) \prec_{\mathsf{A}} (a^*, b^*) \quad \text{and} \quad (a^*, b) \prec_{\mathsf{B}} (a^*, b^*).$$

- A two-person normal form game is called strictly ordinal potential (SOP) if there exists a potential function $\Psi \colon [K] \times [K] \to \mathbb{R}$ such that, for each player $i \in \{\mathsf{A}, \mathsf{B}\}$,

$$\Theta(a, b) \prec_{\mathsf{A}} \Theta(a', b) \iff \Psi(a, b) > \Psi(a', b),$$
$$\Theta(a, b) \prec_{\mathsf{B}} \Theta(a, b') \iff \Psi(a, b) > \Psi(a, b').$$

Ordinal potential games, continued

- W.l.o.g., the potential function can be chosen to take all values $1, \ldots, K^2$.
- The class of such potentials be denoted by $\mathcal{P}_K$.
- We identify the function $\Psi \in \mathcal{P}_K$ with the $[K] \times [K]$ matrix of its values.

Potential function

- Two SOP games having the same potential are strategically equivalent, i.e., they have the same set of NE.
- Any SOP game is strategically equivalent to a game where $\prec_{\mathsf{A}} \equiv \prec_{\mathsf{B}}$.
- The potential identifies the set of NE of any SOP game.
- We identify the equivalence class of strategically equivalent SOPs with their potential $\Psi \in \mathcal{P}_K$.
- Each NE of an SOP game is a local minimum of its potential.
- The set of NE of $\Psi \in \mathcal{P}_K$ will be denoted by NE_K.
- Its cardinality $|\mathsf{NE}_K|$ will be denoted by W_K.
- The set

$$\Psi^* := \{\Psi(\boldsymbol{\eta})\colon \boldsymbol{\eta} \in \mathsf{NE}_K\} \subset [K^2]$$

  is the set of equilibrium potentials.
- The NE in a SOP can be ordered according to their potential, so that the equilibrium $\boldsymbol{\eta}_1$ is the one with the smallest potential, $\boldsymbol{\eta}_2$ is the one with the second smallest potential, etc.
- The ranking of equilibrium $\boldsymbol{\eta}$ is denoted by $\Lambda(\boldsymbol{\eta})$.

Random ordinal potential games

- For every fixed K, we consider a uniform distribution over the set $\mathcal{P}_K$.
- In a random SOP game the number of NE is a random variables with values in $[K^2]$.
- This random number is positive, since the potential always has a global minimum and cannot be larger than K because the preferences are strict, i.e., the values that the potential can take are all distinct.
- Hence, every row or column can contain at most one NE.

Expected number of equilibria

- It is easy to compute the expected number of NE in a SOP.

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- It is easy to compute the expected number of NE in a SOP.
- The probability that a profile is a NE is $1/(2K-1)$.
- Since the game has K^2 profiles, the expected number of NE is

$$\mathrm{E}[W_K] = \frac{K^2}{2K-1}.$$

- This implies

$$\lim_{K\to\infty} \frac{\mathrm{E}[W_K]}{K} = \frac{1}{2}.$$

Concentration

- The number of NE concentrates (Mimun et al. (2024)).

Theorem

For all $\delta > 0$,

$$\lim_{K \to \infty} \mathrm{P}\left(\left| \frac{W_K}{K} - \frac{1}{2} \right| < \delta \right) = 1.$$

Best-response dynamics

- (a_0, b_0) is a starting strategy profile.
- For each $t \geq 0$ $\mathrm{BRD}(t)$ is a process on $[K^{\mathrm{A}}] \times [K^{\mathrm{B}}]$ such that

$$\mathrm{BRD}(0) = (a_0, b_0)$$

and, if $\mathrm{BRD}(t) = (a', b')$, then, for t even,

$$\mathrm{BRD}(t+1) = (a'', b'),$$

where $a' \neq a'' \in \arg\max_{a \in [K^{\mathbf{A}}]} U^{\mathrm{A}}(a, b')$, if such an action a'' exists, otherwise

$$\mathrm{BRD}(t+1) = \mathrm{BRD}(t);$$

for t odd,

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where $b' \neq b'' \in \arg\max_{b \in [K^{\mathbf{B}}]} U^{\mathrm{B}}(a', b)$, if such an action b'' exists, otherwise

$$\mathrm{BRD}(t+1) = \mathrm{BRD}(t).$$

Best-response dynamics, continued

- If for some $\hat{t}$,

$$\text{BRD}(\hat{t}) = \text{BRD}(\hat{t}+1) = \text{BRD}(\hat{t}+2) = (a^*, b^*),$$

  then $\text{BRD}(t) = (a^*, b^*)$ for all $t \geq \hat{t}$ and (a^*, b^*) is a NE of the game.
- The algorithm stops when it visits an action profile for the second time.
- If this profile is the same as the one visited at the previous time, then a NE has been reached.

Best-response dynamics, continued

- Since the game is SOP, a NE is always reached by the BRD.
- A BRD never visits a row or column more than twice (once by the row player and once by the column player), so it reaches a NE in at most $2K$ steps.
- Once a starting point $\mathrm{BRD}(0) = (a_0, b_0)$ is chosen, the BRD will reach (deterministically) one NE.

Basin of attractions

- For each NE (a^*, b^*), we define its basin of attraction (BoA) as follows:

$$\begin{aligned} &\mathrm{BoA}(a^*, b^*) \\ &\quad := \left\{(a, b) \colon \text{if } \ \mathrm{BRD}(0) = (a, b), \ \text{then } \lim_{t \to \infty} \mathrm{BRD}(t) = (a^*, b^*)\right\}. \end{aligned}$$

- Given the way the process $\mathrm{BRD}(\,\cdot\,)$ is defined, we have that $(a, b) \in \mathrm{BoA}(a^*, b^*)$ implies $(a', b) \in \mathrm{BoA}(a^*, b^*)$ for all $a' \in [K]$.

A lemma

Lemma

If the potential Ψ' is obtained from Ψ by permuting rows and columns, then the NE of Ψ' are just the corresponding permutations of the NE of Ψ.

Moreover, the basin of attractions of the NE in Ψ' are obtained by permutating the columns of the corresponding basin of attractions in Ψ.

An example

$$
\begin{array}{cccccccccc}
4 & 1 & 5 & 2 & 2 & 4 & 1 & 1 & 5 & 4
\end{array}
$$

$$
\begin{pmatrix}
100 & 56 & 43 & 32 & 26 & 24 & 12 & 55 & 39 & 40 \\
77 & 83 & 82 & 48 & 29 & 79 & 44 & 92 & 53 & 95 \\
97 & 3 & 28 & 23 & 57 & 30 & 91 & 17 & 41 & 89 \\
21 & 63 & 99 & 73 & 59 & 4 & 25 & 49 & 85 & 9 \\
42 & 66 & 20 & 72 & 27 & 54 & 68 & 98 & 71 & 67 \\
31 & 15 & 6 & 50 & 90 & 18 & 70 & 81 & 84 & 34 \\
96 & 16 & 5 & 38 & 78 & 65 & 47 & 36 & 8 & 60 \\
69 & 64 & 86 & 10 & 2 & 46 & 61 & 35 & 13 & 14 \\
45 & 1 & 62 & 74 & 19 & 52 & 7 & 11 & 51 & 94 \\
37 & 75 & 88 & 80 & 33 & 76 & 22 & 87 & 58 & 93
\end{pmatrix}
$$

An example, continued

1	2	4	5	1	5	4	2	1	4
1	19	52	62	7	51	94	74	11	45
64	2	46	86	61	13	14	10	35	69
3	57	30	28	91	41	89	23	17	97
63	59	4	99	25	85	9	73	49	21
16	78	65	5	47	8	60	38	36	96
15	90	18	6	70	84	34	50	81	31
56	26	24	43	12	39	40	32	55	100
66	27	54	20	68	71	67	72	98	42
75	33	76	88	22	58	93	80	87	37
83	29	79	82	44	53	95	48	92	77

An example, continued

- The potential in the second matrix is obtained by permuting some rows and columns of the potential in the first matrix.
- The green numbers in the above matrices indicate the potential equilibria.
- The numbers above the matrices indicate the potential of the equilibrium to which the column is attracted.

Asymptotics

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- In particular, we will focus on their size.
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Theorem

For all $\varepsilon \in [0, 1/2)$, *we have*

$$\lim_{K \to \infty} \mathsf{E}\left[\frac{1}{K}\left|\mathrm{BoA}_K(\boldsymbol{\eta}_{\lfloor \varepsilon K \rfloor})\right|\right] = \varphi(\varepsilon) := \exp\left\{\sqrt{1 - 2\varepsilon}\right\}.$$

Asymptotic ranking

- The following corollary shows how the ranking Λ of the equilibria reached by the BRD behaves asymptotically.

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Corollary

Let $\mathrm{BRD}_K(0)$ *be chosen uniformly at random on* $K \times K$. *For all* $\varepsilon \in [0, 1/2)$, *we have*

$$\lim_{K \to \infty} \mathrm{P}(\Lambda(\mathrm{BRD}(2K)) \leq \varepsilon K) = \Phi(\varepsilon) := \int_0^{\varepsilon} \exp\left\{\sqrt{1-2u}\right\} \mathrm{d}u.$$

Moreover,

$$\lim_{K \to \infty} \mathrm{E}\left[\frac{\Lambda(\mathrm{BRD}(2K))}{K}\right] = \int_0^{1/2} u \exp\left\{\sqrt{1-2u}\right\} \mathrm{d}u = \mathrm{e} - \frac{5}{2} \approx 0.21.$$

Figures

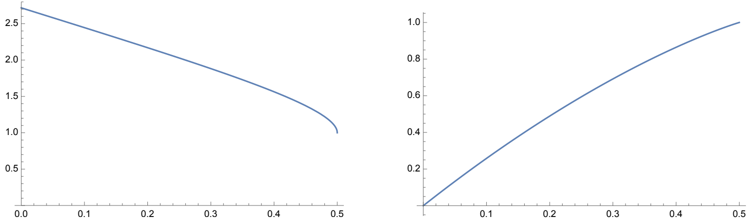

Figure 1: Plot of the functions $\varphi(\cdot)$ (left) and $\Phi(\cdot)$ (right).

continued

- Fig. 1 shows the plot of the function φ and Φ.
- They represent the density and distribution function, respectively, of the basin of attraction of equilibria reached by the BRD, ordered by their ranking.

Theorem

For all $\delta > 0$, *we have*

$$\lim_{K \to \infty} \mathsf{P}\left(\left| \frac{\Psi(\boldsymbol{\eta}_{W_K})}{K \log K} - 1 \right| < \delta \right) = 1.$$

Incremental construction

- The main theorem is proved using what we call the incremental construction of the game.
- This construction provides the potential of a random SOP game that does not have a uniform distribution, but has the same set of equilibrium potentials that a uniformly distributed potential has.

Incremental construction, continued

- For a fixed integer K, we will construct a random potential function $\Psi \in \mathcal{P}_K$ by adding entries sequentially according to the algorithm described below.

For $t \in \left[K^2\right]$,

(a) we call R_t the number of non-empty rows, C_t the number of non-empty columns, and G_t the number of green entries after adding the first t entries of Ψ;

(b) we call M_t the sub-matrix of Ψ composed of rows $[R_t]$ and columns $[C_t]$;

(c) we call $\mathfrak{R}_t$ a Bernoulli random variable such that

$$\mathsf{P}(\mathfrak{R}_t = 1) = \rho_t := \frac{(K - R_{t-1})K}{K^2 - t - 1};$$

(d) we call $\mathfrak{C}_t$ a Bernoulli random variable such that

$$\mathsf{P}(\mathfrak{C}_t = 1) = \kappa_t := \frac{(K - C_{t-1})}{K}.$$

Algorithm

Algorithm 1 Incremental construction

1. Set $\Psi(1,1) = 1$.
2. Color the entry $(1,1)$ green.
3. For $t \in \{1, \ldots, K^2\}$, the $t+1$-th entry is added as follows:
   (a) If $\mathfrak{R}_{t+1} = 1$, then set $R_{t+1} = R_t + 1$.
      (i) If $\mathfrak{C}_{t+1} = 1$, then set $C_{t+1} = C_t + 1$, $\Psi(R_{t+1}, C_{t+1}) = t+1$, and color (R_{t+1}, C_{t+1}) green.
      (ii) If $\mathfrak{C}_{t+1} = 0$, then set $C_{t+1} = C_t$, sample Z_{t+1} uniformly at random in $[C_t]$, and set $\Psi(R_{t+1}, Z_{t+1}) = t+1$.
   (b) If $\mathfrak{R}_{t+1} = 0$, then set $R_{t+1} = R_t$ and draw one entry uniformly at random among the empty entries in the rows $\{1, \ldots, R_t\}$. Call this entry $(X_{t+1}, Y_{t+1}) \in [R_t] \times [K]$.
      (i) If $(X_{t+1}, Y_{t+1}) \notin M_t$, then set $C_{t+1} = C_t + 1$ and $\Psi(X_{t+1}, C_{t+1}) = t+1$.
      (ii) If $(X_{t+1}, Y_{t+1}) \in M_t$, then set $C_{t+1} = C_t$ and $\Psi(X_{t+1}, Y_{t+1}) = t+1$.
4. The output of the algorithm will be called Ψ.

An example, continued

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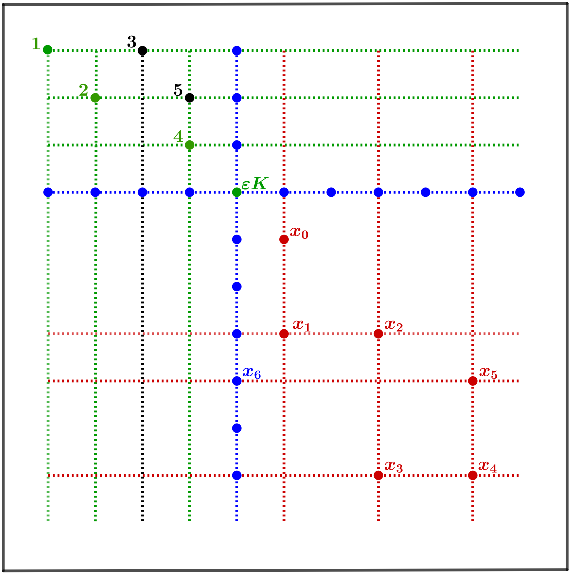

Figure 2

Thank you!